A Proof of the Digit-Concatenation Identity

An original proof, presented with a counting-table argument

Statement

Let $a_1a_2a_3\ldots a_m$ be an m-digit number and let $b_1b_2b_3\ldots b_n$ be an n-digit number. Then the number formed by concatenating the two, $a_1a_2a_3\ldots a_mb_1b_2b_3\ldots b_n$, satisfies:

$$a_1a_2a_3\ldots a_mb_1b_2b_3\ldots b_n \;=\; a_1a_2a_3\ldots a_m\,[\,\underline{00\ldots0}\,,\ n \text{ zeros}] \;+\; b_1b_2b_3\ldots b_n$$

(Here concatenation is written by juxtaposition, and the block in brackets denotes $a_1a_2\ldots a_m$ followed by n zeros.)

Idea of the Proof

The key observation is that if we list, in order, all whole numbers starting from $a_1a_2\ldots a_m00\ldots0$ (n zeros) and stop once we reach $a_1a_2\ldots a_mb_1b_2\ldots b_n$, then the count of "numbers between them, including the ending number," is exactly $b_1b_2\ldots b_n$ itself — because the last n digits simply count upward from $000\ldots0$ to $b_1b_2\ldots b_n$ while the leading block $a_1a_2\ldots a_m$ stays fixed. This is recorded in Table A below.

Table A

Starting number	Ending number	Number of numbers between them and including ending number
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots001$ (n-1 zeros)	1
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots002$ (n-1 zeros)	2
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots003$ (n-1 zeros)	3
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots004$ (n-1 zeros)	4
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots005$ (n-1 zeros)	5
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots006$ (n-1 zeros)	6
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_m00\ldots007$ (n-1 zeros)	7
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	. . .	. . .
$a_1a_2a_3\ldots a_m00\ldots0$ (n zeros)	$a_1a_2a_3\ldots a_mb_1b_2b_3\ldots b_n$	$b_1b_2b_3\ldots b_n$

Table A. Each row keeps the same starting number and lists an ending number reached by advancing the last n digits, together with the count of numbers from just after the start up to and including that ending number.

Key Relation

For any row of Table A, if k denotes the value in the third column (the number of numbers between the starting and ending number, including the ending number), then the ending number is obtained from the starting number simply by adding k:

$$\textbf{ending number} = \textbf{starting number} + k$$

This holds because consecutive whole numbers increase by exactly 1 each step, so counting k numbers forward from the starting number (landing on the k-th one) is the same as adding k to the starting number.

Applying the Relation to the Final Row

Now apply this relation to the last row of Table A, where:

starting number = $a_1a_2a_3\ldots a_m00\ldots0$ (n zeros) | *ending number* = $a_1a_2a_3\ldots a_mb_1b_2b_3\ldots b_n$ | $k = b_1b_2b_3\ldots b_n$

Substituting these into “ending number = starting number + k” gives:

$$a_1a_2a_3\ldots a_mb_1b_2b_3\ldots b_n = a_1a_2a_3\ldots a_m00\ldots0 \text{ (n zeros)} + b_1b_2b_3\ldots b_n$$

which is precisely the identity we set out to prove. ■

End of proof.

Worked Examples

Three numeric illustrations of the digit-concatenation identity

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The three examples below apply the identity proved above to specific numbers, using the same counting-table argument. In each case, the leading block $a_1a_2\ldots a_m$ stays fixed while the last n digits count upward from 0 to $b_1b_2\ldots b_n$, so the number of steps taken equals $b_1b_2\ldots b_n$ itself.

Example 1 — m = 3, n = 2

Let $a_1a_2a_3$ = 123 (m = 3 digits) and b_1b_2 = 45 (n = 2 digits). Concatenating gives 12345, and the identity predicts:

$$12345 = 12300 + 45 = 12300 + 45$$

Table A_1 — counting from 12300 up to 12345

Starting number	Ending number	Number of numbers between them and including ending number
12300	12301	1
12300	12302	2
12300	12303	3
12300	12304	4
12300	12305	5
12300	12306	6
12300	12307	7
12300	. . .	. . .
12300	**12345**	**45**

Table A_1. The last two digits climb from 01 to 45 while "123" is unchanged, so reaching 12345 takes exactly 45 steps from 12300 — giving 12345 = 12300 + 45.

Example 2 — m = 2, n = 3

Let a_1a_2 = 12 (m = 2 digits) and $b_1b_2b_3$ = 345 (n = 3 digits). Concatenating gives 12345, and the identity predicts:

$$12345 = 12000 + 345 = 12000 + 345$$

Table A_2 — counting from 12000 up to 12345

Starting number	Ending number	Number of numbers between them and including ending number
12000	12001	1
12000	12002	2
12000	12003	3
12000	12004	4
12000	12005	5
12000	12006	6
12000	12007	7
12000	. . .	. . .
12000	**12345**	**345**

Table A_2. Here n = 3, so three trailing digits climb from 001 to 345 while "12" is unchanged. Reaching 12345 takes exactly 345 steps from 12000 — giving 12345 = 12000 + 345.

Example 3 — m = 4, n = 1

Let $a_1a_2a_3a_4 = 1234$ (m = 4 digits) and $b_1 = 7$ (n = 1 digit). Concatenating gives 12347, and the identity predicts:

$$12347 = 12340 + 7 = 12340 + 7$$

Table A_3 — counting from 12340 up to 12347

Starting number	Ending number	Number of numbers between them and including ending number
12340	12341	1
12340	12342	2
12340	12343	3
12340	12344	4
12340	12345	5
12340	12346	6
12340	. . .	. . .
12340	**12347**	**7**

Table A_3. With n = 1, only the single last digit climbs, from 1 to 7, while "1234" is unchanged. Reaching 12347 takes exactly 7 steps from 12340 — giving 12347 = 12340 + 7.

In every case the same pattern holds: multiplying the leading block by 10^n (shifting it left to make room for n trailing zeros) and then adding the trailing block reproduces the concatenated number, exactly as the general proof establishes. ■

A Digit-Pattern Proof

$$k\underbrace{00\ldots0}_{n} = k \times 10^n$$

for single-digit k, built step by step

Theorem. For every single-digit number $k \in \{1, 2, \ldots, 9\}$ (extending to $k = 10$),

$$k\underbrace{00\ldots0}_{n} = k \times 1\underbrace{00\ldots0}_{n} = k \times 10^n$$

Section 1 (k = 2)

$$\begin{aligned}
200\ldots(n \text{ zeros}) &= 199\ldots(n \text{ nines}) + 1 \\
1\underbrace{9\ldots9}_{n} &= 1\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 1000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 1 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{2 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 1000\ldots(n \textit{ zeros}) = 1 \times 1000\ldots(n \textit{ zeros})
\end{aligned}$$

Section 2 (k = 3)

$$\begin{aligned}
300\ldots(n \text{ zeros}) &= 299\ldots(n \text{ nines}) + 1 \\
2\underbrace{9\ldots9}_{n} &= 2\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 2000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 2000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 2 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{3 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 2000\ldots(n \textit{ zeros}) = 2 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 1)}
\end{aligned}$$

Section 3 (k = 4)

$$
\begin{aligned}
400\ldots(n \text{ zeros}) &= 399\ldots(n \text{ nines}) + 1 \\
3\underbrace{9\ldots9}_{n} &= 3\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 3000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 3000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 3 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{4 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 3000\ldots(n \textit{ zeros}) = 3 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 2)}
\end{aligned}
$$

Section 4 (k = 5)

$$
\begin{aligned}
500\ldots(n \text{ zeros}) &= 499\ldots(n \text{ nines}) + 1 \\
4\underbrace{9\ldots9}_{n} &= 4\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 4000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 4000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 4 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{5 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 4000\ldots(n \textit{ zeros}) = 4 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 3)}
\end{aligned}
$$

Section 5 (k = 6)

$$
\begin{aligned}
600\ldots(n \text{ zeros}) &= 599\ldots(n \text{ nines}) + 1 \\
5\underbrace{9\ldots9}_{n} &= 5\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 5000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 5000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 5 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{6 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 5000\ldots(n \textit{ zeros}) = 5 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 4)}
\end{aligned}
$$

Section 6 (k = 7)

$$
\begin{aligned}
700\ldots(n \text{ zeros}) &= 699\ldots(n \text{ nines}) + 1 \\
6\underbrace{9\ldots9}_{n} &= 6\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 6000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 6000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 6 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{7 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 6000\ldots(n \textit{ zeros}) = 6 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 5)}
\end{aligned}
$$

Section 7 (k = 8)

$$
\begin{aligned}
800\ldots(n \text{ zeros}) &= 799\ldots(n \text{ nines}) + 1 \\
7\underbrace{9\ldots9}_{n} &= 7\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 7000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 7000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 7 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{8 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 7000\ldots(n \textit{ zeros}) = 7 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 6)}
\end{aligned}
$$

Section 8 (k = 9)

$$
\begin{aligned}
900\ldots(n \text{ zeros}) &= 899\ldots(n \text{ nines}) + 1 \\
8\underbrace{9\ldots9}_{n} &= 8\underbrace{0\ldots0}_{n} + \underbrace{9\ldots9}_{n} \\
&= 8000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1 \\
&= 8000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&\quad \textit{As } 999\ldots(n \textit{ nines}) + 1 = 1000\ldots(n \textit{ zeros}) \\
&= 8 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros}) \\
&= \mathbf{9 \times 1000\ldots(n} \text{ zeros}) \\
&\quad \textit{As } 8000\ldots(n \textit{ zeros}) = 8 \times 1000\ldots(n \textit{ zeros}) \textit{ (Sec 7)}
\end{aligned}
$$

Section 9 (k = 10)

$$1\underbrace{0\ldots0}_{n+1} = \underbrace{9\ldots9}_{n+1}+1$$

$$9\underbrace{9\ldots9}_{n} = 9\underbrace{0\ldots0}_{n}+\underbrace{9\ldots9}_{n}$$

$$= 9000\ldots(n \text{ zeros}) + 999\ldots(n \text{ nines}) + 1$$

$$= 9000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros})$$

As $999\ldots(n\ nines)+1 = 1000\ldots(n\ zeros)$

$$= 9 \times 1000\ldots(n \text{ zeros}) + 1000\ldots(n \text{ zeros})$$

$$= \mathbf{10 \times 1000\ldots(n} \text{ zeros)}$$

As $9000\ldots(n\ zeros) = 9 \times 1000\ldots(n\ zeros)$ *(Sec 8)*

This closes the pattern: $10 \times 10^n = 10^{n+1} = 1\underbrace{0\ldots0}_{n+1}$ ■

Proof: Multiplying by Powers of 10

Why the result always ends in zeros

THEOREM

If **n** is a positive integer and **k** is a positive integer, then multiplying n by 10^k appends exactly k zeros to the right of n's digits.

Proof — Positional (Place-Value) Argument

Every positive integer n can be written in expanded decimal form, as a sum of digits multiplied by powers of 10:

$$n = a_m \cdot 10^m + a_{m-1} \cdot 10^{m-1} + \ldots + a_1 \cdot 10 + a_0$$

where each a_i is a digit from 0–9.

Multiplying both sides by 10^k distributes across every term:

$$n \cdot 10^k = a_m \cdot 10^{m+k} + a_{m-1} \cdot 10^{m-1+k} + \ldots + a_1 \cdot 10^{1+k} + a_0 \cdot 10^k$$

Notice that **every** exponent on the right-hand side is now at least k. This means the lowest place value present is 10^k — so the digit positions 10^0 through 10^{k-1} (the last k digits) must all be zero.

Proof — Repeated Shifting Argument

Since $10^k = 10 \times 10 \times \ldots \times 10$ (k factors), we can view the multiplication as k separate multiplications by 10:

$$n \times 10^k = n \times (10 \times 10 \times \ldots \times 10)$$

Each single multiplication by 10 shifts every digit one place to the left in the decimal system, and fills the newly vacated units place with a 0. Applying this shift k times in a row appends exactly k zeros to the right of n.

QED — both arguments confirm the theorem.

Worked Examples

n	k	n × 10■	Zeros added
7	1	70	1

7	2	700	2
7	3	7,000	3
42	2	4,200	2
123	4	1,230,000	4

Detailed Step-by-Step Examples

Applying the positional-notation proof to real numbers

EXAMPLE — n = 7, k = 3

Step 1 — Write n in expanded (place-value) form:

$n = 7 = 7 \cdot 10^0$

Step 2 — Multiply every term by 10^3, raising each exponent by 3:

$7 \cdot 10^3$ (since 0 + 3 = 3)

Step 3 — Read off the new place values. The smallest exponent present is now 3, so positions $10^{\blacksquare}$ through 10^2 are empty → they become 3 trailing zeros:

Positions 10^0, 10^1, 10^2 are empty → 3 zeros. The digit 7 lands in the thousands place.

Result: $7 \times 10^3 = 7{,}000$

EXAMPLE — n = 42, k = 2

Step 1 — Write n in expanded (place-value) form:

$n = 42 = 4 \cdot 10^1 + 2 \cdot 10^0$

Step 2 — Multiply every term by 10^2, raising each exponent by 2:

$4 \cdot 10^{1+2} + 2 \cdot 10^{0+2} = 4 \cdot 10^3 + 2 \cdot 10^2$

Step 3 — Read off the new place values. The smallest exponent present is now 2, so positions $10^{\blacksquare}$ through 10^1 are empty → they become 2 trailing zeros:

Positions 10^0 and 10^1 are empty → 2 zeros. The original digits 4 and 2 keep their order, just shifted 2 places left.

Result: $42 \times 10^2 = 4{,}200$

EXAMPLE — n = 305, k = 3

Step 1 — Write n in expanded (place-value) form:

$n = 305 = 3 \cdot 10^2 + 0 \cdot 10^1 + 5 \cdot 10^0$

Step 2 — Multiply every term by 10^3, raising each exponent by 3:

$3 \cdot 10^{2+3} + 0 \cdot 10^{1+3} + 5 \cdot 10^{0+3} = 3 \cdot 10^5 + 0 \cdot 10^4 + 5 \cdot 10^3$

Step 3 — Read off the new place values. The smallest exponent present is now 3, so positions $10\blacksquare$ through 10^2 are empty $\rightarrow$ they become 3 trailing zeros:

Positions 10^0, 10^1, 10^2 are empty $\rightarrow$ 3 new trailing zeros. Note the internal zero of 305 is simply carried along — the proof works the same whether or not n itself already contains zeros.

Result: $305 \times 10^3 = 305{,}000$

INTUITION

Multiplying by 10 shifts every digit one place value to the left (ones $\rightarrow$ tens, tens $\rightarrow$ hundreds, and so on). This always leaves an empty ones place, which is filled with a 0. Repeating the shift k times creates k trailing zeros.